Q1→ Prove that $(A\cdot\sigma)(B\cdot\sigma) = (A\cdot B) + i\sigma\cdot(A\times B)$

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by using this relation prove that:-

$$e^{i\theta\sigma\cdot\hat{n}} = \cos\theta + i\sigma\cdot\hat{n}\sin\theta$$

L.H.S. We know that in three dimensions the Pauli & two constants can be written as:-

$$\sigma = i\sigma_x + j\sigma_y + k\sigma_z$$

$$A = iA_x + jA_y + KA_z$$

$$B = iB_x + jB_y + KB_z$$

$$(A\cdot\sigma)(B\cdot\sigma) = [(iA_x + jA_y + KA_z)(i\sigma_x + j\sigma_y + K\sigma_z)][(iB_x + jB_y + KB_z)(i\sigma_x + j\sigma_y + K\sigma_z)]$$

$$= (A_x\sigma_x + A_y\sigma_y + A_z\sigma_z)(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z)$$

$$= A_xB_x\sigma_x^2 + A_yB_y\sigma_y^2 + A_zB_z\sigma_z^2 + A_xB_y\sigma_x\sigma_y + A_xB_z\sigma_x\sigma_z + A_yB_x\sigma_y\sigma_x + A_yB_z\sigma_y\sigma_z + A_zB_x\sigma_z\sigma_x + A_zB_y\sigma_z\sigma_y$$

Now from the property of Pauli Matrix we know that

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$$

and

$$\sigma_x\sigma_y = i\sigma_z \; ; \; \sigma_y\sigma_z = i\sigma_x \; ; \; \sigma_z\sigma_x = i\sigma_y$$

Then above eq^n can be written as :-

$$= A_xB_x + A_yB_y + A_zB_z + i[\sigma_x(A_yB_z - A_zB_y) + \sigma_y(B_zA_x - A_xB_z) + \sigma_z(A_xB_y - A_yB_x)]$$

$= (A\cdot B) + i\sigma\cdot(A\times B)$

Hence,
$$\boxed{(A\cdot\sigma)(B\cdot\sigma) = (A\cdot B) + i\sigma\cdot(A\times B)} \quad \text{— (A)}$$

Now, we know that :-

$$e^{x} = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ldots$$

Also,

$$e^{i\theta\sigma\cdot\hat{n}} = 1 + i\theta(\sigma\cdot\hat{n}) + \frac{[i\theta(\sigma\cdot\hat{n})]^2}{\lfloor 2} + \frac{(i\theta(\sigma\cdot\hat{n}))^3}{\lfloor 3} + \frac{(i\theta(\sigma\cdot\hat{n}))^4}{\lfloor 4} + \frac{(i\theta(\sigma\cdot\hat{n}))^5}{\lfloor 5} + \ldots\ldots$$

$$e^{i\theta\sigma\cdot\hat{n}} = 1 + i\theta(\sigma\cdot\hat{n}) - \frac{\theta^2(\sigma\cdot\hat{n})(\sigma\cdot\hat{n})}{\lfloor 2} - \frac{i\theta^3(\sigma\cdot\hat{n})(\sigma\cdot\hat{n})(\sigma\cdot\hat{n})}{\lfloor 3} + \frac{\theta^4(\sigma\cdot\hat{n})(\sigma\cdot\hat{n})(\sigma\cdot\hat{n})(\sigma\cdot\hat{n})}{\lfloor 4} + \frac{i\theta^5(\sigma\cdot\hat{n})(\sigma\cdot\hat{n})(\sigma\cdot\hat{n})(\sigma\cdot\hat{n})(\sigma\cdot\hat{n})}{\lfloor 5} + \ldots\ldots$$

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Now by eq^n (A) on replacing $A = B = \hat{n}$

$$(\hat{n}\cdot\sigma)(\hat{n}\cdot\sigma) = (\hat{n}\cdot\hat{n}) + i\sigma(\hat{n}\times\hat{n})$$

$$= 1 + i\sigma \times 0 = 1$$

$$e^{i\theta(\sigma\cdot\hat{n})} = 1 + i\theta(\sigma\cdot\hat{n}) - \frac{\theta^2}{\lfloor 2} - \frac{i\theta^3(\sigma\cdot\hat{n})}{\lfloor 3} + \frac{\theta^4}{\lfloor 4} + \frac{i\theta^5(\sigma\cdot\hat{n})}{\lfloor 5} \cdots$$

$$= \left(1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} \cdots\right) +$$

$$i(\sigma\cdot\hat{n})\left(\theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} + \cdots\right)$$

$$\boxed{e^{i\theta(\sigma\cdot\hat{n})} = \cos\theta + i(\sigma\cdot\hat{n})\sin\theta}$$